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SOME OPTIMAL PROBLEMS OF THE THEORY OF LONGITUDINAL VIBRATIONS OF RODS

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Formulations are presented of a number of optimization problems of the theory of longitudinal vibrations of rectilinear rods of constant cross section. Results of their solution, obtained by using the necessary condition of stationarity of the functional constructed in [1] and the necessary Weierstrass condition of a strong minimum of the functional established below, are described. Special attention is paid to optimization problems in which there are discontinuities in the Lagrange multipliers on the characteristic lines on equations of hyperbolic type by which longitudinal vibrations are described.

1. Formulation of the problem. Let us consider the following second order partial differential equation defined in the domain Ω $(0 \leqslant x \leqslant T,\ 0 \leqslant y \leqslant l)$:

$$z_{xx} - w^2 z_{yy} = u_1(x, y) \tag{1.1}$$

If it describes the longitudinal vibrations of a rod, then $z = z(x, y)$ is the longitudinal displacement of a rod section, and $u_1(x, y)$ is the longitudinal load intensity distributed along the rod length. Let us consider the load constrained by the inequality

$$|u_1(x, y)| \leqslant F \tag{1.2}$$

Let be given the initial boundary conditions

$$z(0, y) = z_x(0, y) = 0 \tag{1.3}$$

$$z(x, 0) = 0, \quad z(x, l) = 0 \tag{1.4}$$

We formulate the following optimization problem. To find among the continuous functions $z(x, y)$ and among the piecewise-continuous disturbing forces $u_1(x, y)$ which satisfy (1.1) and the inequality (1.2) in the domain Ω and conditions (1.3) and (1.4) on the boundaries of this domain, those which will render the functional a minimum

$$I = \int_0^l \varphi(z(T, y), z_x(T, y))\, dy \tag{1.5}$$

Various selections of the integrand φ in the integral (1.5) result in diverse problems. Thus, for $\varphi = z_x(T, y)$ we obtain the problem of the minimum of the mean value of the velocity of a section relative to the length of the rod at a finite time $x = T$, and for $\varphi = -z(T, y)$ we have the problem of cumulative disturbances for the longitudinal vibrations of a rod. Other examples of functionals will be presented below.

A singularity of the problems described, which is related to the presence of the inequality (1.2), can be bypassed if an auxiliary real variable $u_2 = u_2(x, y)$ is introduced and the auxiliary dependence

$$\psi = \psi(u_1, u_2) = u_1^2 + u_2^2 - F^2 = 0 \tag{1.6}$$

is constructed. Then the formulated problem can be considered a particular case of the following optimization problem for systems described by second order equations of hyperbolic type.

To find among the continuous functions $z(x, y)$ and the piecewise-continuous equations $u_1(x, y), \ldots, u_m(x, y)$ satisfying the differential equation and relationships

$$L(z) = a_{11}z_{xx} + a_{22}z_{yy} + a_1 z_x + a_2 z_y = f(x, y, z, u) \tag{1.7}$$

$$\psi_k(x, y, u) = 0 \quad (k = 1, \ldots, r < m) \tag{1.8}$$

in the domain Ω $(a \leqslant x \leqslant b,\ c \leqslant y \leqslant d)$, and the conditions

$$z(a, y) = \varphi_1(y), \quad z_x(a, y) = \varphi_2(y) \tag{1.9}$$

$$\varphi_c(x, z, z_y) = 0 \quad \text{for } y = c \tag{1.10}$$

$$\varphi_d(x, z, z_y) = 0 \quad \text{for } y = d$$

on the boundary of the domain Ω, those which render the functional a minimum

$$I = \iint_\Omega f_0(x, y, z, u)\, dx\, dy + \int_c^d \varphi_b(y, z, z_x)\, dy + \chi(z^\circ(b, y)) \tag{1.11}$$

Here $u = (u_1, \ldots, u_m)$ is understood to be an m-dimensional control vector, the coefficients $a_1(x, y)$, $a_2(x, y)$, $a_{11}(x, y)$, $a_{22}(x, y)$ and the functions $f_0, f, \psi_k, \varphi_1, \varphi_2, \varphi_c, \varphi_d, \chi$ are considered continuous and to have continuous partial derivatives with respect to their arguments to the third order inclusive, $z^\circ(b, y)$ denotes the p-dimensional vector $z^\circ(b, y) = (z(b, y_1^\circ), \ldots, z(b, y_p^\circ))$, where y_k° are given numbers and $y_1^\circ = c$, $y_p^\circ = d$, the function φ_b is piecewise-continuous where it has continuous partial derivatives with respect to its arguments to third order inclusive on each continuity section.

Such a problem has been studied in [1], where the necessary condition for the stationarity of the functional I was established. The additional Weierstrass condition of a strong minimum of the functional, obtained below, will be used in solving the optimal problems.

2. Necessary Weierstrass condition of a strong minimum of the functional. The functions $z = z(x, y)$ define a surface in x, y, z space. Let us consider the normal surface E rendering the functional I a minimum. As in the case of one independent variable [2], let us consider the normal surface E to be that surface for which a unique system of Lagrange multipliers exists. In the domain Ω let us select an arbitrary segment M_1M_2 of the line $x = x^\circ = \text{const}$ and the segment M_3M_4 of the line $x = x^\circ + e = \text{const}$, where e is a parameter so that the rectangle $M_1M_2M_3M_4$ formed by these segments would lie entirely in some elementary region ω_i (Fig. 1). Let us draw characteristics C_1, C_{1e}, C_2, C_{2e} of (1.7) through the points $M_1 M_2 M_3$ and M_4. Let Ω_1, Ω_2, Ω_3, Ω_4 and Ω_5 denote the domains obtained. For definiteness, let us consider the characteristic C_2 to intersect the section of the boundary $y = c$ in the domain Ω, and the characteristic C_1 the section of the boundary $x = b$.

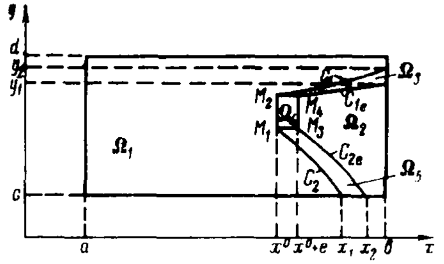

Fig. 1

Let us construct three admissible one-parameter families of surfaces

$$
\begin{aligned}
&z(x, y),\ u_k(x, y)\ (k = 1, \ldots, m), \quad x, y \in \Omega_1 + \Omega_3 + \Omega_5 \\
&Z(x, y),\ U_k(x, y) \quad (k = 1, \ldots, m),\ x, y \in \Omega_4 \\
&z(x, y, e),\ u_k(x, y) \quad (k = 1, \ldots, m),\ x, y \in \Omega_2 \quad |e| \leqslant \varepsilon
\end{aligned}
\tag{2.1}
$$

including E for $e = 0$. The first and third families in (2.1) satisfy (1.7) and (1.8), and the second is constructed by using the equations

$$
L(Z) = f(x, y, Z, U), \quad \psi_k(x, y, U) = 0 \quad (k = 1, \ldots, r) \tag{2.2}
$$

The conditions

$$
\begin{gathered}
z(x^\circ, y) = Z(x^\circ, y), \quad z_x(x^\circ, y) = Z_x(x^\circ, y), \quad y \in [M_1, M_2] \\
Z(x^\circ + e, y) = z(x^\circ + e, y, e), \quad y \in [M_3, M_4] \\
Z_x(x^\circ + e, y) = z_x(x^\circ + e, y, e)
\end{gathered}
\tag{2.3}
$$

are satisfied on the boundaries of the domain. The function $z(x, y)$ and its derivative with respect to the normal are continuous on the characteristics C_1, C_2, C_{1e}, C_{2e} since $z(x, y)$ is continuous in the whole domain Ω and $z_N(x, y)$ is continuous because M_1, M_2, M_3, M_4 lie within the elementary domains ω_i and the characteristics C_1, C_2, C_{1e} and C_{2e} are not boundaries of the elementary domains.

Variations of the family (2.1) with respect to the parameter e on the surface E are defined by the expressions

$$\xi = (\partial z / \partial e)_{e=0} = 0,\ x,\ y \in \Omega_1 + \Omega_3 + \Omega_5$$
$$Z_x(x^\circ, y) = z_x(x^\circ, y, 0) + \xi$$
$$Z_{xx}(x^\circ, y) = z_{xx}(x^\circ, y, 0) + \xi_x, \quad y \in [M_3, M_4] \tag{2.4}$$
$$\zeta_k(x, y) = (\partial u_k / \partial e)_{e=0} = 0, \quad x, y \in \Omega$$

On the basis of (2.3), we have $\xi = 0$ for $y \in [M_3, M_4]$. This variation also equals zero on the characteristics C_{1e} and C_{2e} so that $\xi = 0$. After substitution of (2.1) into the functional I and differentiating with respect to e at $e = 0$ we obtain

$$\left(\frac{dI}{de}\right)_{e=0} = \int_{M_3}^{M_4} [L_2(Z, U) - L_2(z, u)]\, dy + \int_{S_2} \{[a_1\lambda\xi + a_{11}\lambda\xi_x + (a_{11}\lambda)_x \xi]\, n_1 +$$
$$[a_2\lambda\xi + a_{22}\lambda\xi_y - (a_{22}\lambda)_y \xi]\, n_2\}\, ds + \int_{x_1}^{b} \eta_c \left(\frac{\partial\varphi_c}{\partial z}\xi + \frac{\partial\varphi_c}{\partial z_y}\xi_y\right) dx + \tag{2.5}$$
$$\int_c^{y_1} \left(\frac{\partial\varphi_b}{\partial z}\xi + \frac{\partial\varphi_b}{\partial z_x}\xi_x\right) dy + \sum_{\gamma=1}^{p} \frac{\partial\chi}{\partial z_\gamma}\xi(b, y_\gamma^0)$$

where we used the notation

$$L_2 = f_0 + \lambda [L(z) - f] + \sum_{k=1}^{r} \mu_k \psi_k \tag{2.6}$$

and the continuity of this function as it passed across the characteristics C_1 and C_2 is used in the calculation of (2.5). The relations $\varphi_c^- = \varphi_c^+$, $\varphi_b^- = \varphi_b^+$ and the Euler equation are also taken into account, and S_2 denotes the boundary of the domain Ω_2.

Let us go from the variables x, y in (2.5) to the new variables N and s, described in detail in [1], where the formulas and notation used below are also presented. After this passage, we have

$$\left(\frac{dI}{de}\right)_{e=0} = \int_{M_1}^{M_2} [L_2(Z, U) - L_2(z, u)]\, dy + \oint_{S_2} \{A_1\lambda\xi_N - A_2\lambda\xi_s +$$
$$\left[a_1\lambda n_1 + a_2\lambda n_2 - (A_1\lambda)_N - (A_2\lambda)_s - (A_1 - A_3)\frac{1}{\rho}\lambda\right]\xi\Big\}\, ds + \tag{2.7}$$
$$\int_{x_1}^{b} \eta_c \left(\frac{\partial\varphi_c}{\partial z}\xi + \frac{\partial\varphi_c}{\partial z_y}\xi_y\right) dx + \int_c^{y_2} \left(\frac{\partial\varphi_b}{\partial z}\xi + \frac{\partial\varphi_b}{\partial z_x}\xi_x\right) dy + \sum_{\gamma=1}^{p} \frac{\partial\chi}{\partial z_\gamma}\xi(b, y_\gamma^\circ)$$

The contour S_2 consists of the segment M_1M_2 of the two characteristics C_1 and C_2 and segments of the outer boundaries $[c, y_2]$ and $[x_1, b]$. We have $A_1 = 0$, $\xi = 0$ on the characteristics. Hence, $\xi_s = 0$ also. On the outer boundary $A_2 = 0$. Also A_2 and ξ equal zero on the segment M_1M_2, hence ξ_s is also zero. Taking the above into account, we obtain the following expression in place of (2.7)

$$\left(\frac{dJ}{de}\right)_{e=0} = \int_{M_1}^{M_2} [L_2(Z, U) - L_2(z, u)]\, dy - \int_{M_1}^{M_2} A_1\lambda\xi\, dy +$$
$$\int_{x_1}^{b} \left\{\left[\eta_c \frac{\partial\varphi_c}{\partial z} - (a_2 - A_{1y})\lambda + A_1\lambda_y\right]\xi + \left(A_1\lambda - \eta_c\frac{\partial\varphi_c}{\partial z_y}\right)\xi_y\right\} dx +$$

$$\int_{c}^{y_2}\left\{\left[\frac{\partial\varphi_b}{\partial z}+(a_1-A_{1x})\lambda-A_1\lambda_x\right]\xi+\left(A_1\lambda+\frac{\partial\varphi_b}{\partial z_x}\right)\xi_x\right\}dy+\sum_{\gamma=1}^{p}\frac{\partial\chi}{\partial z_\gamma}\xi(b,y_\gamma^\circ) \tag{2.8}$$

The integrands in the third and fourth integrals in the right side of (2.8) are zero on the basis of the boundary conditions. If still another condition at the points $y = y_\gamma^\circ$ is taken into account, then (2.8) can be given the following simple form:

$$\left(\frac{dJ}{de}\right)_{e=0}=\int_{M_1}^{M_2}[L_2(Z,U)-L_2(z,u)-A_1\lambda\xi_x]\,dy \tag{2.9}$$

If the variation of ξ_x from the third equality in (2.4) is substituted into this expression, and it is taken into account that the function $z(x, y)$ is continuous together with its derivatives z_x, z_y and z_{yy} along the line M_1M_2, then the derivative (2.9) can be rewritten as

$$\left(\frac{dI}{de}\right)_{e=0}=\int_{M_1}^{M_2}[H(x,y,z,U,\mu,\lambda)-H(x,y,z,u,\mu,\lambda)]\,dy \tag{2.10}$$

$$H = f_0 - \lambda f + \sum_{k=1}^{r}\mu_k\psi_k$$

The surface E renders the functional I a minimum. Hence

$$(dI\,/\,de)_{e=0} \geqslant 0 \tag{2.11}$$

The arbitrary segment M_1M_2 could be of any length. Therefore, to comply with the inequality (2.11) it is necessary to satisfy the following inequality in each elementary domain ω_t:

$$H(x,y,z,U,\mu,\lambda)-H(x,y,z,u,\mu,\lambda)\geqslant 0 \tag{2.12}$$

Therefore, in order for the surface E to render the functional I a strong minimum, it is necessary to satisfy the stationarity condition and the Weierstrass inequality (2.12) at any of its points, where U is the set of any admissible controls, and u is a set of controls rendering the functional I a minimum, where $U \neq u$.

3. Construction of optimal loadings for continuous Lagrange multipliers. Let us consider some examples of applying the necessary Weierstrass condition described above, starting with problems in which there are no discontinuities in the Lagrange multipliers $\lambda(x, y)$.

As the first illustration, let us construct the loading $u_1(x, y)$ constrained by the inequality (1.2) for a rod, described by (1.1), (1.3) and (1.4) and rendering the functional a maximum

$$I=\frac{2\pi w}{l}\int_0^l z(T,y)\sin\frac{2\pi y}{l}\,dy \tag{3.1}$$

In solving the problem, we introduce the relationship (1.6) in place of the inequality (1.2), and use the equations and conditions described in [1]. We then have the following equations for the factors λ and μ:

$$\lambda_{xx}-w^2\lambda_{yy}=0,\quad -\lambda-2\mu u_1=0,\quad -2\mu u_2=0 \tag{3.2}$$

with the initial and boundary conditions

$$\lambda(T, y) = 0, \ \lambda_x(T, y) = 2\pi w / l \sin 2\pi y / l$$
$$\lambda(x, 0) = \lambda(x, l) = 0$$

Hence the multiplier λ is [3]

$$\lambda(x, y) = -\sin\frac{2\pi w(T-x)}{l}\sin\frac{2\pi y}{l}$$

and therefore $u_1 = \pm F$ everywhere, except on lines on which $\lambda = 0$. These lines are defined by the equalities

$$y^* = \pm\frac{kl}{2} \quad (k = 1, 2, \ldots), \qquad x^* = T \pm \frac{ln}{2w} \quad (n = 1, 2, \ldots) \tag{3.3}$$

where the sign of the control $u_1(x, y)$ is found by using the inequality

$$\lambda u_1 - \lambda U_1 \leqslant 0 \tag{3.4}$$

obtained from the Weierstrass inequality.

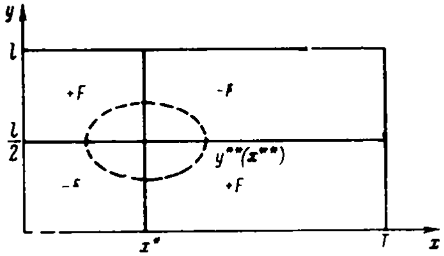

Fig. 2

Shown in Fig. 2 is a distribution of the optimal load $u_1(x, y) = \pm F$ in the domain Ω when $l/2w < T < l/w$. The notation $x^* = T - l/2w$ and $y^* = l/w$ is used. The functional I has the following value:

$$I = \frac{Fl^2}{2\pi w}\left(3 + \cos\frac{2\pi wT}{l}\right)$$

Now, let us examine the problem described above by replacing the functional by the following:

$$I = \frac{2\pi w}{l}\int_0^l z(T, y)\left(\sin\frac{2\pi y}{l} + 2\sin\frac{4\pi y}{l}\right) dy \tag{3.5}$$

Repeating the appropriate computations, we obtain an expression for the multiplier

$$\lambda(x, y) = -\sin\frac{2\pi w(T-x)}{l}\sin\frac{2\pi y}{l} - \frac{1}{2}\sin\frac{4\pi w(T-x)}{l}\sin\frac{4\pi y}{l}$$

The switching lines for the control u_1 are given by (3.3) and the relations

$$y^{**} = \begin{cases} l/2\pi \arccos X, & y \leqslant l/2 \\ l/2\pi(2\pi - \arccos X), & y \geqslant l/2 \end{cases}$$
$$X = \left[-2\cos\frac{2\pi w(T-x^{**})}{l}\right]^{-1}$$

Hence, the domain Ω is now subdivided into the eight subdomains shown in Fig. 2. The sign of $u_1(x, y)$ is found by using the inequality (3.4). Therefore, four subdomains of constant u_1 are obtained for the functional (3.1) and there will be eight such subdomains for the functional (3.5).

4. Optimal loading laws for discontinuities in the Lagrange multipliers. Let us consider the optimization problem for (1.1) and (1.6), conditions (1.3) and (1.4), and the functional

$$I = \int_0^l z_x(T, y)\left(1 + \varepsilon\sin\frac{\pi y}{l}\right) dy \tag{4.1}$$

The equations for λ and μ retain their form (3.2), and the boundary and final conditions are written as

$$\lambda(x, 0) = \lambda(x, l) = 0$$
$$\lambda(T, y) = -1 - \varepsilon \sin \pi y / l, \ \lambda_x(T, y) = 0$$

These conditions show that the multiplier $\lambda(x, y)$ has a discontinuity at the points $x = T, y = 0$ and $x = T, y = l$.

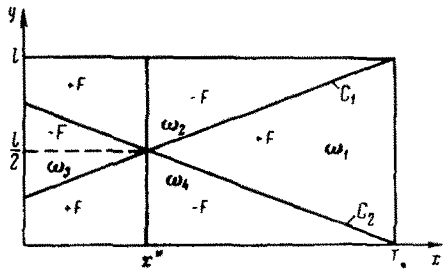

Fig. 3

We seek the solution of the first equation in (3.2) in the form

$$\lambda = \lambda' + \lambda'' \tag{4.2}$$

and require compliance with the conditions

$$\lambda'(T, y) = -1, \lambda''(T, y) = -\varepsilon \sin \pi y / l$$
$$\lambda_x'(T, y) = \lambda_x''(T, y) = 0$$

We shall keep the boundary conditions as before and consider that $l / 2w < T < l / w$.

The solution for λ'' is

$$\lambda''(x, y) = -\varepsilon \cos \frac{\pi w (T - x)}{l} \sin \frac{\pi y}{l} \tag{4.3}$$

In order to find the discontinuous solution $\lambda'(x, y)$, let us draw a characteristic $y = l - w(T - x)$ through the point $x = T, \ y = l$ and a characteristic $y = w(T - x)$ through the point $x = T, y = 0$.

The factor λ' is continuous in the elementary domains being formed (see Fig. 3). The equation defining the discontinuity λ' on the characteristics has the simple form $[\lambda']_s = 0$. Solving it for each of the four segments of the characteristics, we obtain

$$(\lambda_1' - \lambda_2')_{C_1} = -1, \ (\lambda_1' - \lambda_4')_{C_2} = -1$$
$$(\lambda_3' - \lambda_2')_{C_2} = D_1, \ (\lambda_3' - \lambda_4')_{C_1} = D_2$$

The condition [1]

$$\lambda_1' - \lambda_2' + \lambda_3' - \lambda_4' = 0$$

should be satisfied at the point $x = x^*, \ y = l/2$, which results in the values $D_1 = D_2 = 1$. We then have for λ'

$$\lambda'(x, y) = \begin{cases} -1, & x, y \in \omega_1 \\ 0, & x, y \in \omega_2 \\ 0, & x, y \in \omega_4 \\ 1, & x, y \in \omega_3 \end{cases} \tag{4.4}$$

After substituting (4.3) and (4.4) into (4.2), we find that there is just one line $x = x^* = T - {}^1/_2 l / w$ of discontinuity of the control parameter $u_1(x, y)$ for $\varepsilon > 0$. In this case, we have on the basis of the Weierstrass inequality

$$u_1(x, y) = \begin{cases} -F, & x < x^* \\ F, & x > x^* \end{cases}$$

In this case the functional I has the following value:

$$I = I_1 = \frac{Fl^2}{4w}\left(1 + \frac{4w^2 x^{*2}}{l^2}\right) + \frac{2Fl^2\varepsilon}{\pi^2 w}\left(2 - \sin\frac{\pi w T}{l}\right) \tag{4.5}$$

Upon compliance with the inequality $-1 < \varepsilon < 0$, there are in addition to the line already noted, two lines of discontinuity of the parameter $u_1(x, y)$, the characteristics C_1 and C_2. All these lines and the distribution of the values of $u_1(x, y)$ are shown in Fig. 3. For this $u_1(x, y)$ distribution, the functional I has the value

$$I = I_2 = \frac{Fl^2}{4w}\left(1 + \frac{4w^2x^{*2}}{l^2}\right) + \frac{2Fl^2\varepsilon}{\pi^2 w}\left(\frac{\pi wT}{l} - 2 + \sin\frac{\pi wT}{l} + \frac{1}{2}\sin\frac{2\pi wT}{l}\right) \quad (4.6)$$

Comparing expressions (4.5) and (4.6), we find that $I_1 = I_2$ for $\varepsilon = 0$. For $\varepsilon = 0$ the continuous part of $\lambda''(x, y)$ vanishes, and therefore, $\lambda(x, y) = 0$ in the domains ω_2 and ω_4. In the case under consideration there is an innumerable set of solutions yielding the extremum of the functional I for $\varepsilon = 0$. Two have been obtained above artificially.

Now, let us examine a functional of the form

$$I = -\alpha l z\left(T, \frac{l}{2}\right) + \int_0^l z(T, y)\, dy \quad (4.7)$$

while keeping unchanged all the remaining equations and conditions of the problem.

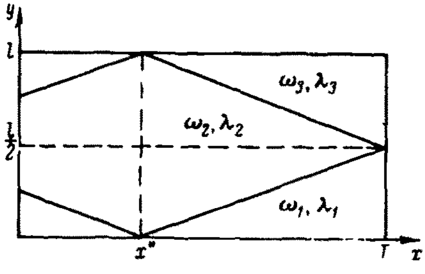

Fig. 4

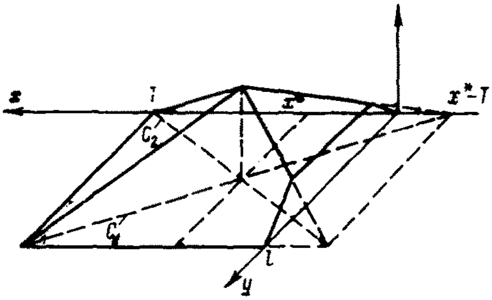

Fig. 5

We draw two characteristics C_1 and C_2 through the point $x = T$, $y = l/2$ (Fig. 4). Then in addition to the boundary conditions $\lambda(x, 0) = \lambda(x, l) = 0$ conditions of the form [1]

$$\lambda_1(T, y) = 0 \quad \text{for} \quad y \in \left[0, \frac{l}{2}\right), \quad \lambda_3(T, y) = 0 \quad \text{for} \quad y \in \left(\frac{l}{2}, l\right]$$

$$\lambda_2\left(T, \frac{l}{2}\right) = \frac{1}{2\sqrt{-a_{11}(T, l/2)\, a_{22}(T, l/2)}}\alpha l$$

$$\lambda_{1x}(T, y) = 1 \quad \text{for} \quad y \in [0, l/2), \quad \lambda_{3x}(T, y) = 1 \quad \text{for} \quad y \in (l/2, l)$$

must be satisfied. Separating the Lagrange multiplier $\lambda(x, y)$ in the sum (4.2) into continuous and discontinuous parts, we obtain that the continuous part can be represented graphically (Fig. 5), where $\lambda''(x^*, l/2) = -T + x^*$. The discontinuous part has the simple form

$$\lambda'(x, y) = \begin{cases} {}^1/_2\, \alpha l / w, & x, y \in \omega_2 \\ 0, & x, y \in \Omega - \omega_2 \end{cases}$$

If α is a small quantity for which the inequality ${}^1/_2 \alpha l / w < l/w - T$ is satisfied, then the lines of discontinuity are disposed as is shown in Fig. 6. Hence $\lambda(x, y) > 0$ for $x, y \in \omega_4, \omega_5, w_6$ and $\lambda(x, y) < 0$ in the remaining domain. The Weierstrass inequality yields

$$u_1(x, y) = \begin{cases} F & \text{for } x, y \in \Omega - \omega_4 - \omega_5 - \omega_6 \\ -F & \text{for } x, y \in \omega_4 + \omega_5 + \omega_6 \end{cases}$$

The functional equals

$$I = \frac{Fl^3}{4w^2}\left[\frac{1}{3} + \frac{wx^*}{l}\left(1 - \frac{4w^2x^{*2}}{3l^2}\right) - \alpha\frac{Fl^3}{2w^2}\left[\frac{1}{4} + \frac{wx^*}{l}\left(1 - \frac{wx^*}{l}\right)\right] + \frac{Fl^3}{4w^2}\alpha^3\right.$$

If $\alpha = 0$, we then arrive at the problem of cumulative disturbances and for

$$T = x_k + \frac{kl}{w}, \quad x_k \in \left[0, \frac{l}{w}\right] \quad (k = 0, 1, \ldots)$$

we obtain

$$I = \frac{Fl^3}{6w^2}\left(k + \frac{3w^2x_k^2}{l^2} - \frac{2w^3x_k^3}{l^3}\right)$$

Let us consider the functional

$$I = \frac{1}{2\varepsilon}\int_{l/2-\varepsilon}^{l/2+\varepsilon} z_x(T, y)\,dy + \alpha\int_0^l z_x(T, y)\sin\frac{\pi y}{l}\,dy \tag{4.8}$$

while keeping the equation and condition of the problem unchanged. Then the equations defining the multipliers λ and μ are written in the form (3.2) and the conditions of the boundary of the domain Ω have the form

$$\lambda(x, 0) = \lambda(x, l) = 0, \quad \lambda_x(T, y) = 0$$

$$\lambda(T, y) = \begin{cases} -\alpha\sin\pi y/l, & y \in [0, l/2 - \varepsilon) \\ -1/\varepsilon - \alpha\sin\pi y/l, & y \in (l/2 - \varepsilon, l/2 + \varepsilon) \\ -\alpha\sin\pi y/l, & y \in (l/2 + \varepsilon, l] \end{cases}$$

Again there are discontinuities in the function λ on the boundary. Let us draw two characteristics through the discontinuous points as is shown in Fig. 7. We seek the multiplier $\lambda(x, y)$ as the sum (4.2) of continuous and discontinuous parts. We then obtain these two parts in the following form:

$$\lambda'(x, y) = \begin{cases} -{}^1/_4\,\varepsilon^{-1}, & x, y \in \omega_2, \omega_4 \\ {}^1/_4\,\varepsilon^{-1}, & x, y \in \omega_6, \omega_8 \\ {}^1/_2\,\varepsilon^{-1}, & x, y \in \omega_3, \\ 0, & \text{in the remaining domain} \end{cases}$$

$$\lambda''(x, y) = -\alpha\cos\frac{\pi w(T - x)}{l}\sin\frac{\pi y}{l}$$

An investigation of the sign of the multiplier $\lambda(x, y) = \lambda'(x, y) + \lambda''(x, y)$ for

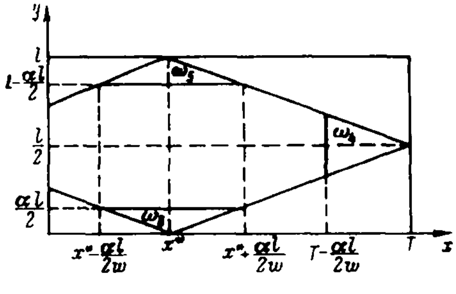

Fig. 6

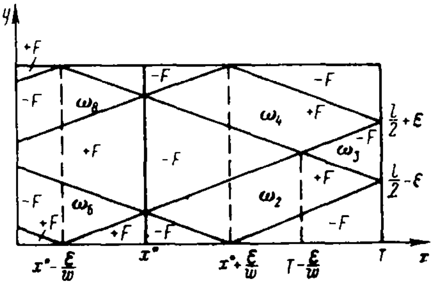

Fig. 7

$\alpha > 0$ and the use of the Weierstrass inequality show that the discontinuity in the function $u_1(x, y)$ holds only for $x = x^*$ in this case, and this function has the form

$$u_1(x, y) = \begin{cases} F, & x > x^* \\ -F, & x < x^* \end{cases}$$

In case $\varepsilon < wx^*$ the functional I equals the following:

$$I = \frac{2Fl^2\alpha}{\pi^2 w}\left(2 - \sin\frac{\pi w T}{l}\right) + FT - \frac{F\varepsilon}{w} \qquad (4.9)$$

Upon compliance with the inequality $-{}^1/_4\,\varepsilon^{-1} < \alpha < 0$ the distribution of the values of the function $u_1(x, y)$ is as shown in Fig. 7. The functional I hence has the value

$$I = \frac{4Fl^2\alpha}{\pi w}\left(\frac{wT}{l} - \frac{\varepsilon}{2l} - \frac{1}{\pi} + \frac{1}{2\pi}\sin\frac{\pi w T}{l} - \frac{1}{2\pi}\sin\frac{2\pi\varepsilon}{l}\right) + FT - \frac{F\varepsilon}{w} \qquad (4.10)$$

Comparing (4.9) and (4.10) for $\alpha = 0$, we obtain the following: $I = FT - F\varepsilon / w$. Let us note that again the problem has an innumerable set of solutions for $\alpha = 0$, two of which are described above.

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PRESSURE OF A STAMP ON A HALF-PLANE WITH INCLUSIONS

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The problem of pressing a stamp on a half-plane with holes in which inclusions from another material are inserted with prestress, is considered. The cases of frictionless contact and for total adhesion of the stamp to the half-plane are examined. It is shown that when the elastic constants of the half-plane and inclusions are identical, the auxiliary functions introduced on the contours are defined completely by the magnitude of the prestress and the solution of the problem is obtained in closed form. If the elastic constants are distinct, then the method proposed results in some functional relationships which can be used to determine the auxiliary functions from the kinematic contact conditions.

1. Formulation of the problem. Let us consider an elastic half-plane S_0 with a finite number of holes. The half-plane is bounded by a line L_0, and the holes